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A Variant of Alaoglu's Theorem for Semicontinuous Functions

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Abstract. Let X be an arbitrary topological space, and $\mathcal{C}(X)$ the convex cone of upper semicontinuous bounded functions on X . Further, let $\mathcal{C}^*(X)$ be its dual, i.e., the convex cone of functionals that are additive, positively homogeneous, and monotone. On $\mathcal{C}^*(X)$, we define the weak* topology as the coarsest topology such that, for any $f \in \mathcal{C}(X)$, the evaluation map $\mu \mapsto \int f d\mu$ is continuous. Then, the unit ball in $\mathcal{C}^*(X)$ is compact in the weak* topology. However, even if X is compact, (i) functionals in $\mathcal{C}^*(X)$ need not be representable as integrals, and (ii) the space of regular Borel probability measures on X may fail to be compact in the weak* topology. In sum, these observations correct a misrepresentation in the literature and show that the standard approach to establishing mixed-strategy equilibrium existence cannot be easily extended to the non-Hausdorff case.

Keywords. Semicontinuous functions · Positively homogeneous, additive, and monotone functionals · Weak* topology · Alaoglu's Theorem · Compactness in convex cones

MSC classification. 46A20 (Duality and reflexivity in topological vector space; 54D30 (Compactness in general topology); 26A48 (Semicontinuous functions)

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1 Introduction

A beautiful theorem attributed to [Alaoglu \(1940\)](#) says that the unit ball in the space adjoint to an arbitrary Banach space is compact in the weak* topology. Especially in the case where the Banach space in question is the space of *continuous* bounded real-valued functions on a topological space X , this result is of substantial applied value. For instance, combined with the Riesz representation theorem ([Dunford and Schwartz, 1958](#), Thm. IV.6.3), it shows that the space of regular Borel probability measures on a compact Hausdorff space is compact in the weak* topology, which is—up to the present day—a crucial step in the derivation of conditions sufficient for the existence of mixed-strategy Nash equilibria in noncooperative games ([Glicksberg, 1952](#); [Fan, 1953](#); [Reny, 1999](#)).¹

In this paper, we investigate the feasibility of developing an analogous theory based on *semicontinuous* functions. There is, in fact, a simple economic rationale for restricting attention to upper or lower semicontinuous functions, because that condition is much tighter than continuity in guaranteeing the existence of a maximum or minimum of an objective function on a compact space (cf., e.g., [Gutiérrez, 2009](#), p. 132). But also from the perspective of topological measure theory ([Pollard, 1976](#); [Bogachev, 2007](#), Ch. 7), the use of semicontinuous functions suggests itself in a natural way when the underlying space is not necessarily Hausdorff, because the system of continuous functions may be “too small” in that case, in the sense that points of the space cannot be separated in general.² For convenience, we will mostly focus on the case of upper semicon-

¹The compactness of the space of regular Borel probability measures on a compact Hausdorff space was long treated as folklore in the game-theoretic literature, until [Glycopantis and Muir \(2004\)](#) provided a detailed proof.

²While the compactness assumption reflects a strong economic intuition ([Bergstrom, 1975](#); [Walker, 1977](#); [Gutiérrez, 2009](#)), the rationale for using the Hausdorff separation axiom, except that it naturally holds in metrizable spaces, has remained less clear. For example, it is known

tinuous (henceforth u.s.c.) functions, noting that the adaptations to the analysis necessary to deal with the case of lower semicontinuous (l.s.c.) functions are of a minor nature only.

We start on the positive side by establishing a variant of Alaoglu’s Theorem. Given a topological space X , we denote by $\mathcal{C}(X)$ the convex cone of real-valued functions on X that are u.s.c. and bounded. We denote by $\mathcal{C}^*(X)$ the *dual cone*, defined as the set of positively homogeneous, additive, and monotone functionals on $\mathcal{C}(X)$. It turns out that any functional in $\mathcal{C}^*(X)$ is both bounded and continuous in the topology induced by the supremum norm. Thus the topological and the algebraic duals coincide in this case. On $\mathcal{C}^*(X)$, we are interested in the weak* topology, i.e., the coarsest topology that renders all evaluation maps, corresponding to u.s.c. and bounded test functions, continuous. We show that the (conical section of the) unit ball in $\mathcal{C}^*(X)$ is compact in the weak* topology. A similar result holds for the unit sphere. These results do not assume that X is Hausdorff.

However, and this is the main point of the present paper, our analogue of Alaoglu’s theorem does not guarantee, even if X is compact, that the space of regular Borel probability measures on X , denoted henceforth by $\mathcal{P}(X)$ is compact in the weak* topology. To explain this point, we first note that a functional in $\mathcal{C}^*(X)$ need not admit a representation as an integral with respect to a Borel measure on X . We present an example (Example 1) of a functional that is not

from [Sion \(1958\)](#) and [Reny \(1999\)](#) that neither the existence of a value nor the existence of pure strategy Nash equilibrium hinges on any separation axiom. In a similar vein, [Balder \(1999, App. A\)](#) extended Kakutani’s fixed point theorem to non-Hausdorff spaces. See also [Goubault-Larrecq \(2018\)](#) and [Khan et al. \(2024\)](#). It is therefore remarkable that, with the sole exception of [Mertens \(1986\)](#), the Hausdorff assumption has been imposed in virtually every analysis of mixed-strategy Nash equilibria.

in the image of a canonical mapping

$$\mathcal{P}(X) \rightarrow \mathcal{C}^*(X). \quad (1)$$

Here, X is chosen as the countably infinite set of the natural numbers, equipped with the cofinite topology, and the functional returns the infimum of any u.s.c. and bounded function on X . Thus, the *Riesz representation property* fails in the example. The discussion seems useful because this point has been misperceived in the literature.

Based on the failure of the Riesz representation property, we deduce the *lack of compactness of the space of regular probability measures*. We advance the discussion of the anomaly above by constructing an explicit instance of an infinite cover of $\mathcal{P}(X)$ that does not admit a finite subcover (Example 2). Each open set $\mathcal{V}_n$ in the cover, for an integer $n \geq 1$, consists of measures that are mixtures giving weight of $\frac{1}{2}$ to some distribution with support contained in $\{1, \dots, n\}$. As $\mathcal{P}(X)$, in this case, corresponds to the set of convergent series with nonnegative terms, this cover has the desired property. These observations constitute a barrier for a generalization of standard existence proofs for mixed strategy Nash equilibria to non-Hausdorff strategy spaces.

The remainder of this paper is organized as follows. Section 2 reviews the related literature. Section 3 contains preliminaries. Section 4 presents our variant of Aloglu's theorem. In Section 5, we show that the Riesz representation property may fail in our setting. In Section 6, we show that the space of regular Borel probability measures need not be compact. Section 7 concludes.

2 Related literature

The proof of our variant of Alaoglu’s theorem follows established routes ([Dunford and Schwartz, 1958](#), V.4.2). Several authors, such as [García-Raffi et al. \(2004\)](#), [Plotkin \(2006\)](#), and [Stonyakin \(2018\)](#), have obtained related results. However, our specific set of assumptions is not covered by those analyses.

The anomaly captured by Example 1 has been documented before by [Topsøe \(1976, Ex. 1\)](#) for the convex cone of nonnegative functions on the integers with well-defined limit, and by [Gignac \(2014, Ex. 3.7 and Cor. 5.5\)](#) for the vector lattice generated by the bounded semicontinuous functions on a non-complete Noetherian space. The present analysis transports the insights contained in those works to the dual of the convex cone of u.s.c. functions on an arbitrary topological space, and thereby underscores the importance of the anomaly for the applied theory of noncooperative games.

Regarding the topological properties of the space of Borel measures, [Gignac \(2014, p. 1782\)](#) wrote that “the measure theory of non-complete spaces lacks the nice properties of the measure theory of complete Noetherian spaces” (such as compactness). However, he did not provide an example (such as our Example 2) illustrating the lack of compactness of $\mathcal{P}(X)$.³

3 Preliminaries

This section introduces the convex cone of upper semicontinuous bounded functions (Subsection 3.1), its dual (Subsection 3.2), and norms on both cones (Subsection 3.3).

³[Mertens \(1986, pp. 243-245\)](#) outlined a “measure theory” for non-Hausdorff compact spaces including a generalization of the Riesz theorem and a proof of the compactness of $\mathcal{P}(X)$. Given the brevity of exposition, however, that article does not provide an immediate answer to the issues explored in the present paper. For further discussion, see Section 7.

3.1 The Convex Cone of U.S.C. Bounded Functions

Let X be an arbitrary topological space (not necessarily Hausdorff). A real-valued function f on X is *upper semicontinuous (u.s.c.)* if, for every $c \in \mathbb{R}$, the upper contour set $\{x \in X : f(x) \geq c\}$ is closed.⁴ We denote by $\mathcal{C}(X)$ the set of u.s.c. and bounded real-valued functions on X . Since $\mathcal{C}(X)$ is closed under finite sums and under multiplication by nonnegative scalars, it forms a *convex cone*. Moreover, there is a canonical injection $\mathbb{R} \hookrightarrow \mathcal{C}(X)$, i.e., $\mathcal{C}(X)$ contains all constants. Let $\mathbf{0}, \mathbf{1} \in \mathcal{C}(X)$ denote the constant functions that are equal to zero and unity on X , respectively. $\mathcal{C}(X)$ is partially ordered in the usual way. For $f, g \in \mathcal{C}(X)$, we write $f \geq g$ if $f(x) \geq g(x)$ for all $x \in X$.

3.2 The Dual of $\mathcal{C}(X)$

A functional $\sigma : \mathcal{C}(X) \rightarrow \mathbb{R}$ is *additive* if $\sigma(f + g) = \sigma(f) + \sigma(g)$, for any $f, g \in \mathcal{C}(X)$. It is *positively homogeneous* if $\sigma(\lambda f) = \lambda \sigma(f)$, for any $f \in \mathcal{C}(X)$ and any $\lambda \geq 0$. A functional $\sigma : \mathcal{C}(X) \rightarrow \mathbb{R}$ is *monotone* if $\sigma(f) \geq \sigma(g)$, for any $f, g \in \mathcal{C}(X)$ such that $f \geq g$.⁵ Let $\mathcal{C}^*(X)$ denote the set of functionals on $\mathcal{C}(X)$ that are additive, positively homogeneous, and monotone. It is again straightforward to check that $\mathcal{C}^*(X)$ is a convex cone, referred to in the sequel as the *dual* of $\mathcal{C}(X)$.

Lemma 1. *Let $c \in \mathbb{R}$ be a constant. Then, $\sigma(-c) = -\sigma(c)$, for any $\sigma \in \mathcal{C}^*(X)$. If, in addition, $\sigma(\mathbf{1}) \leq 1$, then $|\sigma(c)| \leq |c|$.*

Proof. Since $-c \in \mathcal{C}(X)$, additivity and positive homogeneity jointly imply

⁴The *upper topology* on $\mathbb{R}$ has open sets of the form $\emptyset$, $\mathbb{R}$, or $(-\infty, a)$ for $a \in \mathbb{R}$. Then, a function $f : X \rightarrow \mathbb{R}$ is u.s.c. if and only if f is continuous relative to the upper topology (Kelley, 1975, p. 101).

⁵For a positively homogeneous functional $\sigma : \mathcal{C}(X) \rightarrow \mathbb{R}$, we have $\sigma(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the zero function on X . In this case, therefore, monotonicity implies nonnegativity, but the converse is not true in general.

$\sigma(-c) + \sigma(c) = \sigma(\mathbf{0}) = 0$, i.e., $\sigma(-c) = -\sigma(c)$. This proves the first claim. Assume that $\sigma(\mathbf{1}) \leq 1$. Then, $\sigma(-\mathbf{1}) = -\sigma(\mathbf{1}) \geq -1$. Hence, $|\sigma(c)| \leq |c|$, proving the second claim. $\square$

Given any $f \in \mathcal{C}(X)$, we define the functional $f^* : \mathcal{C}^*(X) \rightarrow \mathbb{R}$ via $f^*(\sigma) = \sigma(f)$. On $\mathcal{C}^*(X)$, the *weak* topology* is the coarsest topology such that f^* is continuous for any $f \in \mathcal{C}(X)$. A neighborhood basis for the weak* topology at some point $\sigma \in \mathcal{C}^*(X)$ consists of sets of the form

$$\mathcal{U}_{f_1, \dots, f_K, \varepsilon}(\sigma) = \{\tau \in \mathcal{C}^*(X) : |\tau(f_k) - \sigma(f_k)| < \varepsilon, \forall k \in \{1, \dots, K\}\},$$

where $f_1, \dots, f_K \in \mathcal{C}(X)$ and $\varepsilon > 0$. It is straightforward to check that $\mathcal{C}^*(X)$ is locally convex and Hausdorff.⁶

3.3 Norms on $\mathcal{C}(X)$ and $\mathcal{C}^*(X)$

The usual norms are not needed for the subsequent development. To ease the comparison with the literature, however, we equip $\mathcal{C}(X)$ with the supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

It is then immediate that, for any $f \in \mathcal{C}(X)$, we have $\|f\| = 0$ if and only if $f = 0$. Moreover, $\|rf\| = r\|f\|$ for any $r \in \mathbb{R}_+$ and $f \in \mathcal{C}(X)$. Finally, $\|f + g\| \leq \|f\| + \|g\|$ for any $f, g \in \mathcal{C}(X)$. Thus, $\mathcal{C}(X)$ is a *normed cone* (García-Raffi et al., 2004; Stonyakin, 2018). Given $f \in \mathcal{C}(X)$, the ε -neighborhood of f is defined as the set

$$U_\varepsilon(f) = \{f + g : g \in \mathcal{C}(X) \text{ and } \|g\| < \varepsilon\}.$$

⁶By saying that $\mathcal{C}^*(X)$ is *locally convex*, we mean that each functional has a neighborhood base consisting of convex sets.

We call the topology in which these ε -neighborhoods form a neighborhood basis the *norm topology* on $\mathcal{C}(X)$. We say that a functional $\sigma : \mathcal{C}(X) \rightarrow \mathbb{R}$ is *bounded* if

$$\|\sigma\| = \sup_{f \in \mathcal{C}(X), \|f\|=1} |\sigma(f)| < \infty.$$

Lemma 2. *Any functional $\sigma \in \mathcal{C}^*(X)$ is both bounded and continuous in the norm topology.*

Proof. Take $\sigma \in \mathcal{C}^*(X)$. We will show first that σ is bounded. Take any $f \in \mathcal{C}(X)$ such that $\|f\| = 1$. Then, $-1 \leq f(x) \leq 1$ for all $x \in X$. Hence, $-\mathbf{1} \leq f \leq \mathbf{1}$. By monotonicity and $\sigma(-\mathbf{1}) = -\sigma(\mathbf{1})$, we get $-\sigma(\mathbf{1}) \leq \sigma(f) \leq \sigma(\mathbf{1})$, so that $|\sigma(f)| \leq M \equiv \sigma(\mathbf{1}) < \infty$. Hence, σ is bounded. Next, we show that σ is continuous. By positive homogeneity, $|\sigma(g)| < \varepsilon$ for all $g \in \mathcal{C}(X)$ such that $\|g\| < \delta = \frac{\varepsilon}{M+1}$. Thus, starting from any $f \in \mathcal{C}(X)$ and $\varepsilon > 0$, we find $\delta > 0$ such that $\sigma(f+g) = \sigma(f) + \sigma(g)$ is less than ε apart from $\sigma(f)$. It follows that σ is continuous relative to the norm topology. $\square$

By Lemma 2, each functional has a finite norm, so that $\mathcal{C}^*(X)$ is a normed cone. As usual, the norm topology, which intuitively corresponds to *uniform* convergence, is stronger than the weak* topology, which corresponds to *pointwise* convergence.

4 Modifying Alaoglu's Theorem

We define the *unit ball* in $\mathcal{C}^*(X)$ by

$$\mathcal{B} = \{\sigma \in \mathcal{C}^*(X) : \sigma(\mathbf{1}) \leq 1\}.$$

Given that $\mathcal{C}^*(X)$ is a convex cone, $\mathcal{B}$ is not really a ball geometrically. However, we adopt the term for convenience. The following result is an analogue of Alaoglu's theorem.

Proposition 1. *The unit ball $\mathcal{B}$ is compact in the weak* topology.*

Proof. For any $f \in \mathcal{C}(X)$, the interval $I_f = [\underline{f}, \overline{f}] \subseteq \mathbb{R}$, with $\underline{f} = \inf_{x \in X} f(x)$ and $\overline{f} = \sup_{x \in X} f(x)$, is compact in the usual topology on $\mathbb{R}$. Hence, by Tychonoff's theorem (Kelley, 1975, Thm. 5.13), the product space

$$I = \prod_{f \in \mathcal{C}(X)} I_f$$

is compact as well. For any $f \in \mathcal{C}(X)$, let $\pi_f : I \rightarrow I_f$ denote the canonical projection. Define the subsets

$$A(f, g) = \{\sigma \in I : \pi_{f+g}(\sigma) = \pi_f(\sigma) + \pi_g(\sigma)\},$$

$$B(f, \lambda) = \{\sigma \in I : \pi_{\lambda f}(\sigma) = \lambda \pi_f(\sigma)\}.$$

for all $f, g \in \mathcal{C}(X)$ and $\lambda \geq 0$. Further, let

$$D(f, g) = \{\sigma \in I : \pi_f(\sigma) \geq \pi_g(\sigma)\},$$

for all $f, g \in \mathcal{C}(X)$ such that $f \geq g$. Since these subsets are closed, so is

$$\hat{I} = \left(\bigcap_{f, g \in \mathcal{C}(X)} A(f, g) \right) \cap \left(\bigcap_{f \in \mathcal{C}(X), \lambda \geq 0} B(f, \lambda) \right) \cap \left(\bigcap_{f, g \in \mathcal{C}(X), f \geq g} D(f, g) \right).$$

We now claim that

$$\mathcal{B} = \{\sigma \in \mathcal{C}^*(X) : \sigma(f) \in I_f \text{ for any } f \in \mathcal{C}(X)\}. \quad (2)$$

The inclusion $\supseteq$ is immediate from the fact that $\mathbf{1} \in \mathcal{C}(X)$. Conversely, take $\sigma \in \mathcal{C}^*(X)$ such that $\sigma(\mathbf{1}) \leq 1$. We have to show that $\underline{f} \leq \sigma(f) \leq \overline{f}$, for any $f \in \mathcal{C}(X)$. Take any $f \in \mathcal{C}(X)$. If $\overline{f} = \underline{f}$, then $\sigma(f) = \sigma(\underline{f}) = \underline{f}\sigma(\mathbf{1}) = \underline{f}$. This proves the claim in this case. Suppose, therefore, that $\overline{f} > \underline{f}$. Define $g = (f - \underline{f})/(\overline{f} - \underline{f})$. Note that $\mathbf{0} \leq g \leq \mathbf{1}$. Therefore, $0 \leq \sigma(g) \leq 1$, which implies $\underline{f} \leq \sigma(f) \leq \overline{f}$, as has been claimed. This proves (2). Finally, consider

the mapping $\Phi : \mathcal{B} \rightarrow \widehat{I}$, defined by $\Phi(\sigma) = \sigma(f)$. Then, Φ is a homeomorphism relative to the respective weak* topologies. Hence, $\mathcal{B}$ is indeed compact in the weak* topology. $\square$

A minor but important distinction arises between the *unit ball*, which is the object in Alaoglu's theorem, and the *unit sphere*, which is the subset thereof corresponding in the classical approach to regular probability distributions via the Riesz representation theorem. We define the *unit sphere* in $\mathcal{C}^*(X)$ by

$$\mathcal{S} = \{\sigma \in \mathcal{C}^*(X) : \sigma(\mathbf{1}) = 1\}.$$

By the general properties of initial topologies (Bourbaki, 1966, § 2.3), the subspace topology induced by the weak* topology on the unit sphere is the coarsest topology for which all functionals f^* , restricted to $\mathcal{S}$, are continuous, for any $f \in \mathcal{C}(X)$. We then get the following result.

Corollary 1. *$\mathcal{S}$ is compact in the weak* topology.*

Proof. Both $\mathcal{B} = \{\sigma \in \mathcal{C}^*(X) : \pi_1(\sigma) \leq 1\}$ and $\mathcal{B}' = \{\sigma \in \mathcal{C}^*(X) : \pi_1(\sigma) \geq 1\}$ are weak* closed subsets of $\mathcal{C}^*(X)$. Since $\mathcal{S} = \mathcal{B} \cap \mathcal{B}'$, the claim follows from Proposition 1. $\square$

5 Failure of the Riesz Representation Property

Suppose that X is compact. Then, for any Borel probability measure $\mu \in \mathcal{P}(X)$ and any bounded u.s.c. function $f \in \mathcal{C}(X)$, the Lebesgue integral $\int f d\mu$ is defined as usual. The following example, inspired by Gignac (2014), shows that the canonical mapping (1) from regular Borel probability measures on X to the unit sphere in $\mathcal{C}^*(X)$ is not surjective.

Example 1. Let $X = \mathbb{N}$ be the set of natural numbers, equipped with the cofinite topology. Thus, the open sets are the empty set and complements of arbitrary finite sets. Then, X is compact. $\mathcal{C}(X)$ consists of functions $f: X \rightarrow \mathbb{R}$ of the form $f(x) = c + c_x$, where $c \in \mathbb{R}$ is a constant and independent of x , and $c_x \geq 0$ with $\{x \in X : c_x > \varepsilon\}$ is finite for any $\varepsilon > 0$. Let $\sigma_{\inf} : \mathcal{C}(X) \rightarrow \mathbb{R}$ be the functional defined by $\sigma(f) = c$. Then, $\sigma_{\inf} \in \mathcal{C}^*(X)$. We claim that $\sigma_{\inf}$ cannot be represented as an integral. Indeed, suppose $\sigma_{\inf} = \{f \mapsto \int f d\mu\}$ for some Borel probability measure μ . Then, for each $x \in X$, we have $\mu(\{x\}) = \int_X 1_x d\mu = \sigma_{\inf}(1_x) = 0$, so $\mu = 0$, in contradiction to the assumption that μ is a probability measure.

The “infimum functional” $\sigma_{\inf} \in \mathcal{C}^*(X)$ constructed in the example admits no representing measure $\mu \in \mathcal{P}(X)$, showing that the representation property breaks down in the non-Hausdorff case. Notably, the issue does not lie with the regularity requirement for measures in $\mathcal{P}(X)$. Therefore, the example directly refutes [Fuchssteiner \(1977, Main Thm. & Cor. 4\)](#).⁷ A similar error occurs in [Fuchssteiner and Lusky \(1981, p. 211\)](#).

6 Failure of the Compactness of $\mathcal{P}(X)$

As we are going to show now, $\mathcal{P}(X)$ need not be compact in the weak* topology.

Example 2. We build on [Example 1](#). For $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \{\mu \in \mathcal{P}(X) : \int \delta_{\{1, \dots, n\}} d\mu > \tfrac{1}{2}\},$$

where $\delta_{\{1, \dots, n\}} \in \mathcal{C}(X)$ is the indicator function of the closed set $\{1, \dots, n\} \subseteq X$.

The collection $\{\mathcal{V}_n\}_{n=1}^{\infty}$ is an open cover of $\mathcal{P}(X)$ in the weak* topology. However,

⁷Indeed, in view of [Example 1](#), the step in the proof of [Fuchssteiner \(1977, p. 179\)](#) that deduces “ $\tilde{\mu}_n \leq \sup_{M_n}$ on E ” seems too strong.

no finite subset of it covers $\mathcal{P}(X)$. Thus, $\mathcal{P}(X)$ indeed lacks compactness in the weak* topology.

7 Concluding Discussion

The analysis implies that the weak* topology, as defined above, is, in general, too strong to ensure that the space of regular Borel probability measures on a non-Hausdorff compact space is compact. A natural topology on $\mathcal{C}^*(X)$ even weaker than the weak* topology has been considered by [Topsøe \(1970\)](#). Specifically, this is the coarsest topology for which the mapping $\mu \mapsto \int f d\mu$ is u.s.c., for any $f \in \mathcal{C}(X)$.⁸ Clearly, with fewer sets being defined as open compared to the weak* topology assumed in Proposition 1, both the unit ball and the unit sphere are compact in Topsøe topology. However, in the non-Hausdorff case, no rigorous proof seems available for [Mertens' \(1986\)](#) assertion that $\mathcal{P}(X)$ is compact in that topology, making it hard to judge the suitability of that topology for our specific game-theoretic purpose.⁹ These questions must therefore be left for future work.

References

- Alaoglu, L. (1940). Weak topologies of normed linear spaces. *Annals of Mathematics*, 41(1):252–267. [2](#)
- Balder, E. J. (1999). On the existence of Cournot–Nash equilibria in continuum games. *Journal of Mathematical Economics*, 32(2):207–223. [3](#)
- Bergstrom, T. C. (1975). Maximal elements of acyclic relations on compact sets. *Journal of Economic Theory*, 10:403–404. [2](#)

⁸In fact, this topology coincides with the coarsest topology for which the mapping $\mu \mapsto \int f d\mu$ is l.s.c., for any l.s.c. and bounded function $f : X \rightarrow \mathbb{R}$.

⁹For instance, while the sets $\mathcal{V}_n$ constructed in Example 2 fail to be open in Topsøe's topology, this obviously does not prove that $\mathcal{P}(X)$ is compact in the Topsøe topology.

- Bogachev, V. I. (2007). *Measure Theory, Volume II*. Springer, Berlin. [2](#)
- Bourbaki, N. (1966). *General Topology. Parts 1 & 2*. Elements of Mathematics. Hermann / Addison-Wesley Publishing Company, Paris; Reading, Massachusetts. Translation of *Topologie Générale*. Part 1: vii+437 pp.; Part 2: viii+363 pp. [10](#)
- Dunford, N. and Schwartz, J. T. (1958). *Linear Operators, Part 1: General Theory*. John Wiley & Sons, Hoboken, NJ. [2](#), [5](#)
- Fan, K. (1953). Minimax theorems. *Proceedings of the National Academy of Sciences*, 39(1):42–47. [2](#)
- Fuchssteiner, B. (1977). When does the Riesz representation theorem hold? *Archiv der Mathematik*, 28(1):173–181. [11](#)
- Fuchssteiner, B. and Lusky, W. (1981). *Convex Cones*, volume 56. Elsevier. [11](#)
- García-Raffi, L., Romaguera, S., Sánchez-Pérez, E. A., and Valero, O. (2004). Metrizability of the unit ball of the dual of a quasi-normed cone. *Bollettino dell’Unione Matematica Italiana*, 7:483–492. [5](#), [7](#)
- Gignac, W. (2014). Measures and dynamics on Noetherian spaces. *Journal of Geometric Analysis*, 24(4):1770–1793. [5](#), [10](#)
- Glicksberg, I. L. (1952). A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. *Proceedings of the American Mathematical Society*, 3(1):170–174. [2](#)
- Glycopantis, D. and Muir, A. (2004). The compactness of $Pr(K)$. *Advances in Mathematical Economics*, 6:39–53. [2](#)

- Goubault-Larrecq, J. (2018). A non-Hausdorff minimax theorem. *Minimax Theory and its Applications*, 3(1):73–80. [3](#)
- Gutiérrez, J. M. (2009). A characterization of compactness through preferences. *Mathematical Social Sciences*, 57(1):131–133. [2](#)
- Kelley, J. L. (1975). *General Topology*. Courier Dover, Mineola, NY. [6](#), [9](#)
- Khan, M. A., McLean, R. P., and Uyanik, M. (2024). On constrained generalized games with action sets in non-locally-convex and non-Hausdorff topological vector spaces. *Journal of Mathematical Economics*, 111:102964. [3](#)
- Mertens, J.-F. (1986). The minmax theorem for u.s.c.-l.s.c. payoff functions. *International Journal of Game Theory*, 15(4):237. [3](#), [5](#), [12](#)
- Plotkin, G. D. (2006). A domain-theoretic Banach–Alaoglu theorem. *Mathematical Structures in Computer Science*, 16(2):299–311. [5](#)
- Pollard, D. B. (1976). *Topological Measure Theory, with Applications to Probability*. PhD thesis, Australian National University, Canberra, Australia. [2](#)
- Reny, P. J. (1999). On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, 67(5):1029–1056. [2](#), [3](#)
- Sion, M. (1958). On general minimax theorems. *Pacific Journal of Mathematics*, (8):171–176. [3](#)
- Stonyakin, F. S. (2018). On sublinear analogs of weak topologies in normed cones. *Mathematical Notes*, 103(5):859–864. [5](#), [7](#)
- Topsøe, F. (1970). *Topology and Measure*. Springer, New York, NY. [12](#)

Topsøe, F. (1976). Further results on integral representations. *Studia Mathematica*, 55(3):239–245. [5](#)

Walker, M. (1977). On the existence of maximal elements. *Journal of Economic Theory*, 16(2):470–474. [2](#)